AN INEQUALITY FOR THE DISTRIBUTION OF NUMBERS FREE OF SMALL PRIME FACTORS

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Abstract

Let $1 < z \leq x$ be arbitrary real numbers, and denote by $\Phi(x, z)$ the number of positive integers up to x whose prime divisors are all greater than z. In this note we prove the sharp inequality $\Phi(x, z) < x/\log z$ for all $1 < z \leq x$, improving upon the classical sieve bound $\Phi(x, z) \ll x/\log z$.

1. The Result

One of the fundamental problems in sieve theory is the estimation of

$$S(\mathcal{A}, \mathcal{P}, z) = \#\{n \in \mathcal{A}\colon\ \gcd(n, P(z)) = 1\},$$

where $\mathcal{A} \subseteq \mathbb{N}$ is a subset of positive integers, $\mathcal{P}$ is a subset of primes, $z > 1$ is a positive real number, and

$$P(z) := \prod_{p \in \mathcal{P} \cap [1,z]} p.$$

When $\mathcal{A} = \mathbb{N} \cap [1, x]$ and $\mathcal{P}$ is the set of all primes, where $x \geq z$ is a real number, the quantity $S(\mathcal{A}, \mathcal{P}, z)$ yields the number of positive integers up to x whose prime divisors are all greater than z. Throughout this paper we shall denote this quantity by $\Phi(x, z)$, namely,

$$\Phi(x, z) = \sum_{\substack{n \leq x \\ p \mid n \Rightarrow p > z}} 1.$$

By the inclusion-exclusion principle we have the explicit formula

$$\Phi(x, z) = \sum_{d \mid P(z)} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor,$$

where $\lfloor a \rfloor$ denotes the integer part of a for any $a \in \mathbb{R}$ and μ is the Möbius function. This is the starting point of the sieve of Eratosthenes. When z is fairly small in comparison with x but tending to infinity, say $z = x^{o(1)}$, it is easy to see, by the fundamental lemma of either Brun's sieve [7, Theorem 2.5] or Selberg's sieve [7, Theorem 7.2], together with a classical theorem of Mertens [9, Theorem 429], that

$$\Phi(x,z) \sim x \prod_{p \le z} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma} x}{\log z}, \tag{1}$$

where $\gamma = 0.57721...$ is Euler's constant.

Around 70 years ago, Buchstab and de Bruijn studied the distribution of uncancelled elements in the sieve of Eratostenes. Setting $u := \log x / \log z$ so that $z = x^{1/u}$, Buchstab [3] showed that for any fixed $u > 1$,

$$\Phi(x,z) \sim x e^{\gamma} \omega(u) \prod_{p \le z} \left(1 - \frac{1}{p}\right) \sim \frac{\omega(u) x}{\log z}$$

as $x \to \infty$, where $\omega\colon [1,\infty) \to (0,\infty)$ is the Buchstab function which is defined as the unique continuous solution to the delay differential equation

$$\frac{d}{du}(u\omega(u)) = \omega(u-1), \quad u \ge 2,$$

subject to the initial condition $\omega(u) = 1/u$ for $1 \le u \le 2$. It is convenient to extend the definition of $\omega(u)$ by setting $\omega(u) := 0$ for $u < 1$. Comparing this result with (1) we see that the asymptotic behavior of $\Phi(x,z)$ is somewhat irregular. Maier [11] gave an interesting application of Buchstab's result to the distribution of primes in short intervals. Using the fact that $\omega(u) - e^{-\gamma}$ changes sign in every interval of length one, he showed that for any given $\lambda > 1$ one has

$$\limsup_{x\to\infty} \frac{\pi(x + (\log x)^{\lambda})}{(\log x)^{\lambda-1}} > 1,$$
$$\liminf_{x\to\infty} \frac{\pi(x + (\log x)^{\lambda})}{(\log x)^{\lambda-1}} < 1,$$

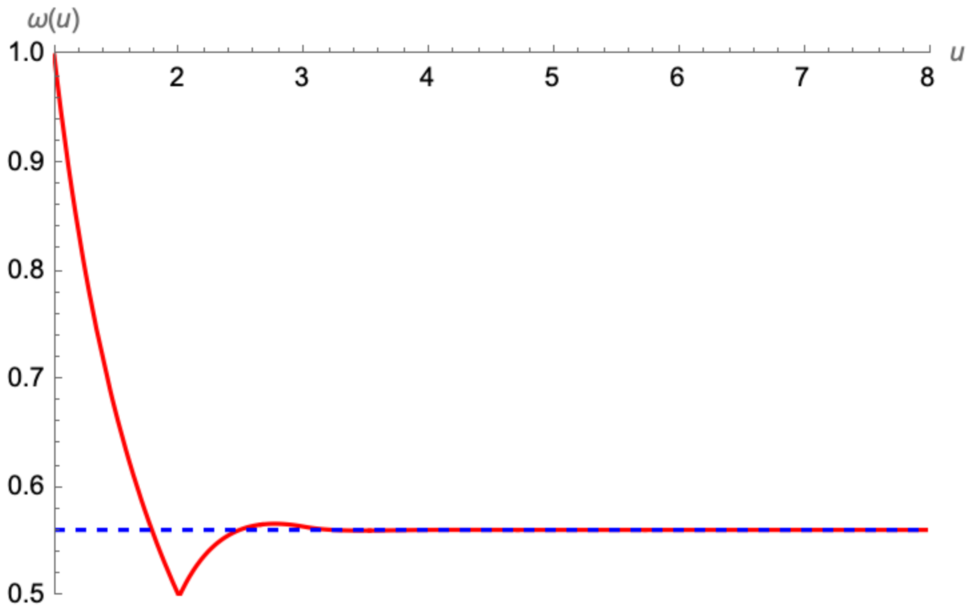
where $\pi(x)$ is the prime counting function. Building on Buchstab's work, de Bruijn [1] showed, among other things, that $\omega(u) \to e^{-\gamma}$ as $u \to \infty$ and that

$$\Phi(x,z) = \mu_z(u) e^{\gamma} x \log z \prod_{p \le z} \left(1 - \frac{1}{p}\right) + O(x \exp(-(\log z)^{3/5-\epsilon})) \tag{2}$$

for $x \ge z \ge 2$, where $\epsilon > 0$ is any given real number and

$$\mu_z(u) := \int_0^{\infty} \omega(u-v) z^{-v}\, dv.$$

In fact, $\omega(u)$ converges to $e^{-\gamma}$ quite rapidly, as one can see from the graph of $\omega(u)$ below generated by Mathematica.



Indeed, Buchstab [3] showed that

$$|\omega(u) - e^{-\gamma}| \leq \frac{\rho(u-1)}{u}$$

for all $u \geq 1$, where $\rho(u)$ is the Dickman-de Bruijn function which is defined to be the unique continuous solution to the delay differential equation

$$u\rho'(u) + \rho(u-1) = 0, \quad u > 1,$$

with the initial condition $\rho(u) = 1$ for $0 < u \leq 1$. Combined with an estimate of de Bruijn on $\rho(u)$ [2, Equ (1.8)] this shows that $|\omega(u) - e^{-\gamma}|$ is

$$\leq \exp\left(-u\left(\log u + \log\log u - 1 + \frac{\log\log u - 1}{\log u} + O\left(\left(\frac{\log\log u}{\log u}\right)^2\right)\right)\right).$$

Finer results on the asymptotic behavior of $\Phi(x, z)$ have been found by Tenenbaum. The reader is referred to his book [14, Chapter III.6] for detailed discussions on this subject.

In the present note we are interested in upper bounds for $\Phi(x, z)$ that are applicable in wide ranges. For instance, a theorem of Hall on the distribution of the mean values of multiplicative functions [8] allows us to obtain upper bounds when z and x are sufficiently large. To state his result, let f denote a multiplicative function such that $0 \le f(n) \le 1$ for all $n \in \mathbb{N}$, and define

$$\Theta(f,x) := \prod_{p \le x} \left(1 - \frac{1}{p}\right) \left(\sum_{k=0}^{\infty} \frac{f(p^k)}{p^k}\right).$$

Hall [8] showed that

$$\frac{1}{x} \sum_{n \le x} f(n) \le e^{\gamma} \left(1 + O\left(\frac{\log\log x}{\log x}\right)\right) \Theta(f,x). \tag{3}$$

This result was later improved by Hildebrand [10], and Granville and Soundararajan [5, 6] have developed a method that connects the mean values of complex multiplicative functions taking values in the closed unit disk with the continuous solutions to certain integral equations. Taking f to be the characteristic function of the set of $n \in \mathbb{N}$ with $\gcd(n, P(z)) = 1$, we obtain at once from (3) that for any fixed $\epsilon > 0$,

$$\Phi(x,z) \le e^{\gamma} x \left(1 + O\left(\frac{\log\log x}{\log x}\right)\right) \prod_{p \le z} \left(1 - \frac{1}{p}\right) < (1+\epsilon) \frac{x}{\log z}$$

for sufficiently large z. The object of this note is to establish the following theorem which shows that the above inequality, with the term ϵ discarded, holds uniformly for all $1 < z \le x$.

Theorem. *For any* $1 < z \le x$ *we have*

$$\Phi(x,z) < \frac{x}{\log z}. \tag{4}$$

Moreover, we have

$$\Phi(x,z) < \frac{x}{\log z} - \frac{x}{2\log^2 z} \tag{5}$$

when $z \ge \max(3, \sqrt{x})$ *and*

$$\Phi(x,z) < \frac{2x}{3\log z} \tag{6}$$

when $\max(2, x^{2/5}) \le z \le \sqrt{x}$.

Proof. The case $1 < z < 2$ is trivial, since

$$\Phi(x) \le x < \frac{x}{\log 2} < \frac{x}{\log z}.$$

For $2 \le z < 3$ we have

$$\Phi(x,z) \le \frac{x+1}{2} < \frac{x}{\log 3} < \frac{x}{\log z}.$$

This proves (4) for $1 < z < 3$. To prove (6) for $2 \le z < 3$, note that $x \ge 4$ and that

$$\Phi(x,z) \le \frac{x+1}{2} < \frac{2x}{3\log 3} < \frac{2x}{3\log z}$$

when $x \ge 5$. Moreover,

$$\Phi(x,z) = 2 < \frac{2x}{3\log 3} < \frac{2x}{3\log z}$$

when $4 \le x < 5$. Hence (6) holds for $2 \le z < 3$.

From now on we shall suppose that $z \ge 3$. Put $y := x^{2/5}$. By the arithmetic large sieve [4, Corollary 9.9] we have

$$\Phi(x,z) \le \frac{x+z^2}{S(z)},$$

where

$$S(z) := \sum_{n \le z} \frac{\mu(n)^2}{\varphi(n)}$$

and φ is Euler's totient function. Montgomery and Vaughan [12, Lemma 7] showed that $S(z) \ge \log z + 1.07$ for all $z \ge 6$. Combining this with the precise values of $S(z)$ for $3 \le z < 6$, we find that $S(z) \ge \log z + 0.89$ for all $z \ge 3$. Thus we have

$$\Phi(x,z) \le \frac{x+z^2}{\log z + 0.89} < \frac{x}{\log z}$$

whenever $z^2 \log z < 0.89x$, which is easily seen to be true when $3 \le z \le y$. This proves (4) in the range $3 \le z \le y$.

Now we treat the case $z \ge \sqrt{x}$. By Theorem 1 and the associated Corollary 1 in [13] we have

$$\Phi(x,z) = 1 + \pi(x) - \pi(z) < 1 + \frac{x}{\log x} + \frac{3x}{2\log^2 x} - \frac{z}{\log z}$$

whenever $x \ge 289$. We compute

$$\frac{\partial}{\partial z}\left(\frac{x+z}{\log z} - \frac{x}{2\log^2 z}\right) = \frac{(z\log z - x)(\log z - 1)}{z\log^3 z}.$$

This implies that as $z \in [3, x]$ varies, the quantity

$$\frac{x+z}{\log z} - \frac{x}{2\log^2 z}$$

is minimized at $z = z_0$, where $z_0 \in [3, x]$ satisfies the equation $z_0 \log z_0 = x$. Hence

$$\frac{x+z}{\log z} - \frac{x}{2\log^2 z} \ge z_0 + \frac{z_0}{2\log z_0} > z_0 + \frac{z_0 \log z_0}{2\log^2(z_0 \log z_0)}.$$

Now (5) will follow if we can show

$$1+\frac{z_0\log z_0}{\log(z_0\log z_0)}+\frac{3z_0\log z_0}{2\log^2(z_0\log z_0)}\le z_0+\frac{z_0\log z_0}{2\log^2(z_0\log z_0)}.$$

Simple computation shows that this is equivalent to

$$(\log\log z_0-1)\log z_0+\log^2\log z_0-\frac{\log^2(z_0\log z_0)}{z_0}\ge 0,$$

which clearly holds if $\log\log z_0\ge 1$ and $(\sqrt{z_0}-1)\log\log z_0\ge\log z_0$. Since $\sqrt{t}-1>\log t$ for all $t\ge e^e$, it suffices to have $z_0\ge e^e$. Using $z_0\log z_0=x$ we see that this is indeed the case when $x\ge 289$. It follows that (5) holds when $x\ge 289$. For $x<289$ we verify using Mathematica that

$$1+\pi(n)-k<\frac{n}{\log p_{k+1}}\left(1-\frac{1}{2\log p_{k+1}}\right)$$

holds for all integers $9\le n<289$ and all $p_k\in[\sqrt{n},n)$, where p_k is the kth prime. This implies that (5) holds for all $3\le x<289$ and $z\ge\sqrt{x}$. We have thus proved (5) in the range $3\le z\le\sqrt{x}$, and hence (4) in the same range.

It remains to consider the range $y\le z\le\sqrt{x}$, where we necessarily have $x\ge 9$. We first prove (4) in this range. We assume that $x\ge 1{,}024$ so that $\sqrt{x}\ge 32$. For $1{,}024\le x<5{,}800$ we have $y\ge 16$ and $1/\log z>0.230$. It follows by the inclusion-exclusion principle that

$$\Phi(x,z)<x\prod_{p\le 11}\left(1-\frac{1}{p}\right)+16<0.208x+16<0.230x<\frac{x}{\log z}$$

for all $1{,}024\le x<5{,}800$. Suppose now that $x\ge 5{,}800$. Since $y^3>x$, any positive integer $n\le x$ with all of its prime divisors greater than z must have at most 2 prime divisors. Hence

$$\Phi(x,z)=1+\pi(x)-\pi(z)+\sum_{z<p\le\sqrt{x}}\ \sum_{p\le q\le x/p}1. \tag{7}$$

We have by [13, Theorem 1] that

$$\pi(x)<\frac{x}{\log x}+\frac{3x}{2\log^2 x}$$

and

$$\sum_{z<p\le\sqrt{x}}\ \sum_{p\le q\le x/p}1\le\sum_{z<p\le\sqrt{x}}\pi\left(\frac{x}{p}\right)\le\sum_{z<p\le\sqrt{x}}\left(\frac{x/p}{\log(x/p)}+\frac{3x/p}{2\log^2(x/p)}\right).$$

Note that $1/(1-t) \le 1 + 2t$ for all $t \in [0, 1/2]$. Thus we have

$$\sum_{z<p\le\sqrt{x}} \frac{x/p}{\log(x/p)} \le \frac{x}{\log x} \sum_{z<p\le\sqrt{x}} \frac{1}{p} + \frac{2x}{\log^2 x} \sum_{z<p\le\sqrt{x}} \frac{\log p}{p}. \tag{8}$$

From Theorem 5 and its corollary in [13] it follows that

$$\sum_{z<p\le\sqrt{x}} \frac{1}{p} \le \log \frac{\log\sqrt{x}}{\log y} + \frac{1}{\log^2\sqrt{x}} + \frac{1}{2\log^2 y} = \log\frac{5}{4} + \frac{57}{8\log^2 x}.$$

By Theorem 6 and its corollary in [13] we find that

$$\sum_{z<p\le\sqrt{x}} \frac{\log p}{p} \le \log\sqrt{x} - \log y + \frac{1}{\log\sqrt{x}} + \frac{1}{2\log y} = \frac{1}{10}\log x + \frac{13}{4\log x}.$$

Inserting these estimates in (7) gives

$$\Phi(x,z) < 1 + \frac{6x}{5\log x} + \frac{3x}{2\log^2 x} + \frac{13x}{2\log^3 x} + \left(\frac{x}{\log x} + \frac{6x}{\log^2 x}\right)\left(\log\frac{5}{4} + \frac{57}{8\log^2 x}\right).$$

Therefore, (4) will follow if we can show

$$1 + \frac{3x}{2\log^2 x} + \frac{13x}{2\log^3 x} + \left(\frac{x}{\log x} + \frac{6x}{\log^2 x}\right)\left(\log\frac{5}{4} + \frac{57}{8\log^2 x}\right) \le \frac{4x}{5\log x},$$

which is equivalent to

$$\frac{\log x}{x} + \frac{3}{2\log x} + \frac{13}{2\log^2 x} + \left(1 + \frac{6}{\log x}\right)\left(\log\frac{5}{4} + \frac{57}{8\log^2 x}\right) \le \frac{4}{5}.$$

Since the left side is a strictly decreasing function of $x \in [e, \infty)$, it is not hard to see by direct calculation that the inequality above holds for all $x \ge 5{,}800$. We have thus shown that (4) holds when $y \le z \le \sqrt{x}$ with $x \ge 1{,}024$. For $9 \le x < 1{,}024$, we confirm using Mathematica that (4) always holds in the range $y \le z \le \sqrt{x}$, finishing the proof of (4).

The proof of (6) is similar but slightly more complicated. We assume first that $x \ge 137{,}550$ so that $x^{2/5} > 113.6$. Note that

$$\Phi(x,z) = 1 + \pi(x) + \pi(\sqrt{x}) - 2\pi(z) + \sum_{z<p\le\sqrt{x}} \sum_{p<q\le x/p} 1. \tag{9}$$

By Theorem 1 and its corollaries in [13] we obtain

$$\pi(x) + \pi(\sqrt{x}) < \frac{x}{\log x} + \frac{3x}{2\log^2 x} + \frac{5\sqrt{x}}{2\log x}$$

and

$$\sum_{z<p\le\sqrt{x}}\ \sum_{p<q\le x/p} 1 = \sum_{z<p\le\sqrt{x}} \left(\pi\left(\frac{x}{p}\right) - \pi(p)\right)$$
$$\le \sum_{z<p\le\sqrt{x}} \left(\frac{x/p}{\log(x/p)} + \frac{3x/p}{2\log^2(x/p)} - \frac{p}{\log p}\right).$$

From [13, Theorems 5,6] it follows that

$$\sum_{z<p\le\sqrt{x}} \frac{1}{p} \le \log\frac{\log\sqrt{x}}{\log z} + \frac{1}{2\log^2\sqrt{x}} + \frac{1}{2\log^2 y} = \log\frac{\log\sqrt{x}}{\log z} + \frac{41}{8\log^2 x}, \tag{10}$$

$$\sum_{z<p\le\sqrt{x}} \frac{\log p}{p} \le \log\frac{\sqrt{x}}{z} + \frac{1}{2\log\sqrt{x}} + \frac{1}{2\log y} = \log\frac{\sqrt{x}}{z} + \frac{9}{4\log x}. \tag{11}$$

Inserting these inequalities in (8) we obtain

$$\sum_{z<p\le\sqrt{x}} \frac{x/p}{\log(x/p)} \le \frac{x}{\log x}\left(\log\frac{\log\sqrt{x}}{\log z} + \frac{41}{8\log^2 x}\right) + \frac{2x}{\log^2 x}\left(\log\frac{\sqrt{x}}{z} + \frac{9}{4\log x}\right).$$

For $z \in [y, \sqrt{x}]$, define

$$u(x,z) := \sum_{z<p\le\sqrt{x}} \left(\frac{3x/p}{2\log^2(x/p)} - \frac{p}{\log p}\right).$$

We claim that $u(x,z) \le 0$ for all $z \in [y, \sqrt{x}]$. To prove this, let

$$h(x,t) := 3x - \frac{2t^2(\log x - \log t)^2}{\log t}$$

for $t \in [y, \sqrt{x}]$. Then

$$\frac{\partial}{\partial t} h(x,t) = -\frac{2t((2\log t - 1)(\log x - \log t) - 2\log t)(\log x - \log t)}{\log^2 t} < 0$$

for all $t \in [y, \sqrt{x})$, since the inequalities $\log t > 1$ and $\log x - \log t \ge \log\sqrt{x} > 2$ imply that

$$(2\log t - 1)(\log x - \log t) - 2\log t > (\log\sqrt{x} - 2)\log t > 0.$$

Hence as a function of t, $h(x,t)$ is strictly decreasing on $[y, \sqrt{x}]$. Note that

$$h(x,y) = 3x^{4/5}\left(x^{1/5} - \frac{3}{5}\log x\right) > 0,$$
$$h(x,\sqrt{x}) = x(3 - \log x) < 0.$$

Thus there exists a unique $t_0 \in (y, \sqrt{x})$ such that $h(x,t) > 0$ for all $t \in [y, t_0)$ and $h(x,t) < 0$ for all $t \in (t_0, \sqrt{x}]$. This implies that as a function of z,

$$u(x,z) = \sum_{z<p\le\sqrt{x}} \frac{h(x,p)}{2p\log^2(x/p)}$$

is decreasing on $[y, t_0]$ and increasing on $[t_0, \sqrt{x}]$. It follows that

$$u(x,z) \le \max\{u(x,y), u(x,\sqrt{x})\}$$

for all $z \in [y, \sqrt{x}]$. It is trivial that $u(x,\sqrt{x}) = 0$. Since $1/(1-t)^2 \le 1 + 6t$ for all $t \in [0, 1/2]$, we have

$$\sum_{y<p\le\sqrt{x}} \frac{3x/p}{2\log^2(x/p)} \le \frac{3x}{2\log^2 x} \sum_{y<p\le\sqrt{x}} \frac{1}{p} + \frac{9x}{\log^3 x} \sum_{y<p\le\sqrt{x}} \frac{\log p}{p}.$$

Combining this with (10) and (11) gives

$$\sum_{y<p\le\sqrt{x}} \frac{3x/p}{2\log^2(x/p)} \le \left(\frac{3}{2}\log\frac{5}{4} + \frac{9}{10}\right)\frac{x}{\log^2 x} + \frac{447x}{16\log^4 x}. \tag{12}$$

Applying partial summation and appealing to Theorem 1 and the affiliated Corollary 2 in [13], we have

$$\begin{aligned}\sum_{y<p\le\sqrt{x}} \frac{p}{\log p} &= \frac{\sqrt{x}}{\log\sqrt{x}}\pi(\sqrt{x}) - \frac{y}{\log y}\pi(y) - \int_y^{\sqrt{x}} \frac{\log t - 1}{\log^2 t}\pi(t)\,dt \\ &> \frac{\sqrt{x}}{\log\sqrt{x}}\left(\frac{\sqrt{x}}{\log\sqrt{x}} + \frac{\sqrt{x}}{2\log^2\sqrt{x}}\right) - \frac{5y^2}{4\log^2 y} - \frac{5}{4}\int_y^{\sqrt{x}} \frac{t(\log t - 1)}{\log^3 t}\,dt.\end{aligned}$$

Integrating by parts we obtain

$$\int_y^{\sqrt{x}} \frac{t(\log t - 1)}{\log^3 t}\,dt = \int_y^{\sqrt{x}} \frac{t}{\log t}\,d\left(\frac{t}{\log t}\right) = \frac{1}{2}\left(\frac{x}{\log^2\sqrt{x}} - \frac{y^2}{\log^2 y}\right).$$

It follows that

$$\sum_{y<p\le\sqrt{x}} \frac{p}{\log p} > \frac{3x}{2\log^2 x} + \frac{4x}{\log^3 x} - \frac{125x^{4/5}}{32\log^2 x} > \frac{1.45x}{\log^2 x},$$

since $0.05\log x > 0.59$ and

$$\frac{0.05x}{\log^2 x} + \frac{4x}{\log^3 x} - \frac{125x^{4/5}}{32\log^2 x} > \left(4.59x^{1/5} - \frac{125}{32}\log x\right)\frac{x^{4/5}}{\log^3 x} > 0$$

for all $x \geq 137{,}550$. Together with (12) this allows us to conclude that

$$u(x,y) < \left(\frac{3}{2}\log\frac{5}{4} - 1.45 + \frac{9}{10} + \frac{447}{16\log^2 x}\right)\frac{x}{\log^2 x} < 0.$$

This proves our claim that $u(x,z) \leq 0$ for all $z \in [y, \sqrt{x}]$. Consequently, we obtain by collecting the estimates above and using (9) that

$$\Phi(x,z) < 1 + \frac{x}{\log x} + \frac{3x}{2\log^2 x} + \frac{5\sqrt{x}}{2\log x} + \frac{xf(x)}{\log x} - g(x,z),$$

where

$$f(x) := \log\frac{5}{4} + \frac{41}{8\log^2 x} + \frac{2}{\log x}\left(\frac{1}{10}\log x + \frac{9}{4\log x}\right) = \log\frac{5}{4} + \frac{1}{5} + \frac{77}{8\log^2 x}$$

and

$$g(x,z) := \frac{x}{\log x}\log\frac{\log z}{\log y} + \frac{2x}{\log^2 x}\log\frac{z}{y}.$$

Observe that

$$\frac{\partial}{\partial z}\left(\frac{2x}{3\log z} + g(x,z)\right) = \frac{x}{3z\log^2 z}Q\left(\frac{\log z}{\log x}\right) > 0$$

for $z \in [y, \sqrt{x}]$, where $Q(t) := 6t^2 + 3t - 2$. Thus we have

$$\frac{2x}{3\log z} + g(x,z) \geq \frac{2x}{3\log y} = \frac{5x}{3\log x}.$$

Therefore, (6) will follow if we can show

$$\frac{\log x}{x} + \frac{3}{2\log x} + \frac{77}{8\log^2 x} + \frac{5}{2\sqrt{x}} \leq \frac{7}{15} - \log\frac{5}{4}.$$

This holds for all $x \geq 137{,}550$. Finally, we verify using Mathematica that

$$1 + \pi(n) + \pi(\sqrt{n}) - 2k + \sum_{i=k+1}^{\pi(\sqrt{n})}\left(\pi\left(\frac{n}{p_i}\right) - i\right) < \frac{2n}{3\log p_{k+1}}$$

holds for all integers $9 \leq n < 137{,}550$ and all $p_k \in [n^{2/5}, \sqrt{n}]$. Hence (6) holds in the range $x^{2/5} \leq z \leq \sqrt{x}$.

This completes the proof of our theorem. □

We remark that (4) is best possible in the sense that for any fixed $\epsilon > 0$, it is not true that $\Phi(x,z) < (1-\epsilon)x/\log z$ holds uniformly in the entire range $1 < z \leq x$. But when $2 \leq z \leq \sqrt{x}$, we expect that a somewhat stronger result holds, as we have seen when $y \leq z \leq \sqrt{x}$. It is possible to modify the proof to obtain (6) for all $\max(2, \sqrt[3]{x}) \leq z \leq \sqrt{x}$. Applying this and the large sieve to the inequality [14, Equation (III.6.17)]

$$\Phi(x,z) \leq \Phi(x, \sqrt[3]{x}) + \sum_{z<p\leq \sqrt[3]{x}} \Phi\left(\frac{x}{p}, p\right) + \frac{x}{\lfloor z \rfloor},$$

we may deduce that $\Phi(x,z) < 0.8x/\log z$ holds uniformly in the range $\max(2, \sqrt[4]{x}) \leq z \leq \sqrt[3]{x}$. However, it is perhaps even true that the stronger inequality $\Phi(x,z) < 0.6x/\log z$ holds uniformly in the range $2 \leq z \leq \sqrt{x}$, and a proof of this may require careful application of sieves in certain explicit forms. By examining the final part of the proof, we see that $\Phi(x,z) < Cx/\log z$ holds in the range $y \leq z \leq \sqrt{x}$ when x is sufficiently large, where

$$C > \frac{12}{25} + \frac{2}{5}\log\frac{5}{4} = 0.56925\ldots$$

is fixed but arbitrary. Furthermore, if M denotes the maximum value of $\omega(u)$ for $u \geq 2$, then de Bruijn's result (2) implies that for any fixed $\epsilon > 0$, we have

$$(1/2 - \epsilon)\frac{x}{\log z} < \Phi(x,z) < (M+\epsilon)\frac{x}{\log z}$$

for all sufficiently large $z \leq \sqrt{x}$. It is easy to see that

$$\omega(u) = \frac{\log(u-1)+1}{u}$$

for all $u \in [2,3]$. According to Mathematica, we should expect $M = 0.56713\ldots$ attained by $\omega(u)$ at $u = 2.76322\ldots$ (which is the solution to the equation $(u-1)\log(u-1) = 1$). It is perhaps true that for every $\epsilon > 0$, the inequality $\Phi(x,z) < (M+\epsilon)x/\log z$ holds in the range $2 \leq z \leq \sqrt{x}$ for all sufficiently large x depending on the choice of ϵ only.

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